INDIAN FOREST SERVICE P (EXAM)-2014

C–HENT–N–LBSTA

MATHEMATICS

Paper—I

Time Allowed : Three Hours

Maximum Marks : 200

QUESTION PAPER SPECIFIC INSTRUCTIONS

Please read each of the following instructions carefully before attempting questions :

There are **EIGHT** questions in all, out of which **FIVE** are to be attempted.

Question no. **1** and **5** are compulsory. Out of the remaining **SIX** questions, **THREE** are to be attempted selecting at least **ONE** question from each of the two Sections A and B.

Attempts of questions shall be counted in chronological order. Unless struck off, attempt of a question shall be counted even if attempted partly. Any page or portion of the page left blank in the answer book must be clearly struck off.

All questions carry equal marks. The number of marks carried by a question/part is indicated against it.

Answers must be written in **ENGLISH** only.

Unless otherwise mentioned, symbols and notations have their usual standard meanings.

Assume suitable data, if necessary and indicate the same clearly.

SECTION—A

Q. 1(a) Show that $u_1 = (1, -1, 0)$, $u_2 = (1, 1, 0)$ and $u_3 = (0, 1, 1)$ form a basis for $\mathbb{R}^3$. Express (5, 3, 4) in terms of u_1, u_2 and u_3. 8

Q. 1(b) For the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. Prove that $A^n = A^{n-2} + A^2 - I, \ \ n \geq 3$. 8

Q. 1(c) Show that the function given by

$$f(x) = \begin{cases} \dfrac{x(e^{1/x} - 1)}{(e^{1/x} + 1)}, & x \neq 0 \\ 0 & , x = 0 \end{cases}$$

is continuous but not differentiable at $x = 0$. 8

Q. 1(d) Evaluate $\iint\limits_R y \dfrac{\sin x}{x} \, dx \, dy$ over R where $R = \{(x, y) : y \leq x \leq \pi/2, \ 0 \leq y \leq \pi/2\}$. 8

Q. 1(e) Prove that the locus of a variable line which intersects the three lines :

$$y = mx,\ z = c;\quad y = -mx,\ z = -c;\quad y = z,\ mx = -c$$

is the surface $y^2 - m^2x^2 = z^2 - c^2$. **8**

Q. 2(a) Let $B = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$. Find all eigen values and corresponding eigen vectors of B viewed as a matrix over :

(i) the real field R

(ii) the complex field C. 10

Q. 2(b) If $xyz = a^3$ then show that the minimum value of $x^2 + y^2 + z^2$ is $3a^2$. 10

Q. 2(c) Prove that every sphere passing through the circle $x^2 + y^2 - 2ax + r^2 = 0,\ z = 0$ cut orthogonally every sphere through the circle $x^2 + z^2 = r^2,\ y = 0$. 10

Q. 2(d) Show that the mapping $T : V_2(\overline{R}) \to V_3(\overline{R})$ defined as $T(a, b) = (a + b, a - b, b)$ is a linear transformation. Find the range, rank and nullity of T. 10

Q. 3(a) Examine whether the matrix $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ is diagonalizable. Find all eigen values.

Then obtain a matrix P such that $P^{-1} AP$ is a diagonal matrix. 10

Q. 3(b) A moving plane passes through a fixed point (2, 2, 2) and meets the coordinate axes at the points A, B, C, all away from the origin O. Find the locus of the centre of the sphere passing through the points O, A, B, C. 10

Q. 3(c) Evaluate the integral

$$I = \int_0^{\infty} 2^{-ax^2}\, dx$$

using Gamma function. 10

Q. 3(d) Prove that the equation :

$$4x^2 - y^2 + z^2 - 3yz + 2xy + 12x - 11y + 6z + 4 = 0$$

represents a cone with vertex at (–1, –2, –3). 10

Q. 4(a) Let f be a real valued function defined on [0, 1] as follows :

$$f(x) = \begin{cases} \dfrac{1}{a^{r-1}}, & \dfrac{1}{a^r} < x \le \dfrac{1}{a^{r-1}},\ r = 1, 2, 3 \ldots\ldots \\ 0 & x = 0 \end{cases}$$

where a is an integer greater than 2. Show that $\int_0^1 f(x)\, dx$ exists and is equal to $\dfrac{a}{a+1}$. 10

Q. 4(b) Prove that the plane ax + by + cz = 0 cuts the cone yz + zx + xy = 0 in perpendicular lines if $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$. 10

Q. 4(c) Evaluate the integral $\iint_R \frac{y}{\sqrt{x^2+y^2+1}}\,dx\,dy$ over the region R bounded between $0 \le x \le \frac{y^2}{2}$ and $0 \le y \le 2$. 10

Q. 4(d) Consider the linear mapping $F : \mathbb{R}^2 \to \mathbb{R}^2$ given as $F(x, y) = (3x + 4y, 2x - 5y)$ with usual basis.

Find the matrix associated with the linear transformation relative to the basis $S = \{u_1, u_2\}$ where $u_1 = (1, 2)$, $u_2 = (2, 3)$. 10

SECTION—B

Q. 5(a) Solve the differential equation :

$$y = 2px + p^2 y,\ p = \frac{dy}{dx}$$

and obtain the non-singular solution. 8

Q. 5(b) Solve :

$$\frac{d^4y}{dx^4} - 16y = x^4 + \sin x.$$ 8

Q. 5(c) A particle whose mass is m, is acted upon by a force $m\mu\left(x+\frac{a^4}{x^3}\right)$ towards the origin. If it starts from rest at a distance 'a' from the origin, prove that it will arrive at the origin in time $\frac{\pi}{4\sqrt{\mu}}$. 8

Q. 5(d) A hollow weightless hemisphere filled with liquid is suspended from a point on the rim of its base. Show that the ratio of the thrust on the plane base to the weight of the contained liquid is $12:\sqrt{73}$. 8

Q. 5(e) For three vectors show that :

$$\bar{a}\times(\bar{b}\times\bar{c})+\bar{b}\times(\bar{c}\times\bar{a})+\bar{c}\times(\bar{a}\times\bar{b})=0.$$ 8

Q. 6(a) Solve the following differential equation :

$$\frac{dy}{dx}=\frac{2y}{x}+\frac{x^3}{y}+x\tan\frac{y}{x^2}.$$ 10

Q. 6(b) An engine, working at a constant rate H, draws a load M against a resistance R. Show that the maximum speed is H/R and the time taken to attain half of this speed is $\frac{MH}{R^2}\left(\log 2-\frac{1}{2}\right)$. 10

Q. 6(c) Solve by the method of variation of parameters :

$$y'' + 3y' + 2y = x + \cos x.$$ 10

Q. 6(d) For the vector $\overline{A}=\frac{x\hat{i}+y\hat{j}+z\hat{k}}{x^2+y^2+z^2}$ examine if $\overline{A}$ is an irrotational vector. Then determine ϕ such that $\overline{A}=\nabla\phi$. 10

Q. 7(a) A solid consisting of a cone and a hemisphere on the same base rests on a rough horizontal table with the hemisphere in contact with the table. Show that the largest height of the cone so that the equilibrium is stable is $\sqrt{3}$ × radius of hemisphere. 15

Q. 7(b) Evaluate $\iint_S \nabla\times\overline{A}\cdot\overline{n}\,dS$ for $\overline{A}=(x^2+y-4)\,\hat{i}+3xy\,\hat{j}+(2xz+z^2)\,\hat{k}$ and S is the surface of hemisphere $x^2 + y^2 + z^2 = 16$ above xy plane. 15

Q. 7(c) Solve the D.E. :

$$\frac{d^3y}{dx^3}-3\frac{d^2y}{dx^2}+4\frac{dy}{dx}-2y=e^x+\cos x.$$ 10

Q. 8(a) A semi circular disc rests in a vertical plane with its curved edge on a rough horizontal and equally rough vertical plane. If the coeff. of friction is μ, prove that the greatest angle that the bounding diameter can make with the horizontal plane is :

$$\sin^{-1}\left(\frac{3\pi}{4}\,\frac{\mu+\mu^2}{1+\mu^2}\right).$$ 15

Q. 8(b) A body floating in water has volumes V_1, V_2 and V_3 above the surface when the densities of the surrounding air are ρ_1, ρ_2, ρ_3 respectively. Prove that :

$$\frac{\rho_2-\rho_3}{V_1}+\frac{\rho_3-\rho_1}{V_2}+\frac{\rho_1-\rho_2}{V_3}=0.$$ 10

Q. 8(c) Verify the divergence theorem for $\overline{A}=4x\hat{i}-2y^2\hat{j}+z^2\hat{k}$ over the region $x^2 + y^2 = 4$, $z = 0$, $z = 3$. 15